Propositional adequacy, independence, and incommensurability

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Abstract

I define a relation of *propositional adequacy* between sentences of a language and sets of interpretations of that language, whereby a language is propositionally adequate just if the language has a basis from which it generates a formula to represent each set of interpretations of the language. This leads to an investigation of how many bases there might be for a language, and the conditions under which a language has such a basis. I end with an open question involving the Generalized Continuum Hypothesis.

1 Introduction

This paper concerns a relation between formal languages and the class of their interpretations. The relation that will concern us is a close relative of the concept of *material adequacy*. A sentential language $\mathscr{L}$ is materially adequate if and only if for each n-placed truth-function f, there is a formula φ composed of the truth-functional connectives of $\mathscr{L}$ and n atomic propositions $p_0, \cdots, p_{n-1}$ such that for any valuation v, $v(\varphi) = T$ if and only if $f(v(p_0), \cdots, v(p_{n-1})) = T$.

The relation of our concern, *propositional adequacy*, is best explained by first looking at the case of material adequacy. Let $\mathscr{L}$ be a sentential language with just two atomic sentences and a materially adequate set of truth-functional connectives. Thus, as there are 16 binary truth-functions, for each such truth-function f_i, there is a formula φ_i using 2 atomic propositions $p(= \varphi_0)$ and $q(= \varphi_1)$, representing that truth-function. We may represent this by the following table.

	p	q	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	φ_{12}	φ_{13}	φ_{14}	φ_{15}
v_0	T	T	T	T	T	T	T	T	F	F	F	F	F	F	F	F
v_1	T	F	T	T	F	T	F	F	F	F	F	T	T	T	T	F
v_2	F	T	T	T	F	F	T	F	F	F	T	T	F	F	T	T
v_3	F	F	T	F	F	T	T	T	F	T	T	T	F	T	F	F

The column on the far left simply indicates a name for valuations of the set $\{p, q\}$: v_0 is the valuation in which both p and q are true, etc.

Note now that, once we've put it like this, we may represent each truth-function as a subset of the set of valuations of $\mathscr{L}$: to take a few examples, the truth-function represented by p can equally be represented by the set $\{v_0, v_1\}$, the function represented by φ_8 may be represented by $\emptyset$, and the function represented by φ_9 may be represented by $\{v_3\}$. This way of seeing things puts the focus, not on truth-functions, but on sets of valuations of $\mathscr{L}$. As such, we may think of each subset of $\{v_0, v_1, v_2, v_3\}$ as expressing a proposition of sorts: if $S \subseteq \{v_0, v_1, v_2, v_3\}$, then S "says" that the truth-values of p and q are those of one of the valuations in S. Thus, for example, φ_{10} represents the proposition that p is false (and q is either true or false), which is equally represented by the set $\{v_2, v_3\}$. In this way, we may call these sets *propositions over* $\{p, q\}$ (combinations of truth-possibilities for p and q) by talking about subsets of the set of valuations of $\{p, q\}$. Such a name is not without philosophical precedent, as here propositions are conceived of combinations of ways the world could be—in other words, as sets of interpretations of the sentences of a language.[1]

So, take it that, instead of representing binary truth-functions, the φs represent instead subsets of $\{v_0, v_1, v_2, v_3\}$. As such, we might say that the set $\{p, q\}$ represents, $\{\{v_0, v_1\}, \{v_0, v_2\}\}$, is a *basis* for $\mathscr{P}(\{v_0, v_1, v_2, v_3\})$: for any $S \subseteq \{v_0, v_1, v_2, v_3\}$, we can represent S using only truth-functional connectives in $\mathscr{L}$ and the elements of $\{p, q\}$.

Note that the reason the set represented by $\{p, q\}$ can act as a basis is because p and q are mutually *independent*: that is, for each subset of $\{p, q\}$ there is a valuation whereon just those members of that subset are true. Equivalently, letting p represent $\{v_0, v_1\}$ and q represent $\{v_1, v_3\}$, the set $\{p, q\} = \{\{v_0, v_1\}, \{v_0, v_2\}\}$ is independent because for any subset S of

[1] It finds its inspiration in L. Wittgenstein's [3, Q.QQQ]: QQQ.

$\{\{v_0, v_1\}, \{v_0, v_2\}\}$, there is a unique valuation v in all and only the members of S.

But once we consider independence in this equivalent way, as a property of sets of subsets of $\{v_0, v_1, v_2, v_3\}$, it becomes apparent that there is more than one independent set: in addition to that set represented by $\{p, q\}$ (that is, $\{\{v_0, v_1\}, \{v_0, v_2\}\}$) there are also those sets represented by, to name only two, $\{\varphi_{10}, \varphi_{14}\}$ (that is, $\{\{v_2, v_3\}, \{v_1, v_2\}\}$), and $\{p, \varphi_7\}$ (that is, $\{\{v_0, v_1\}, \{v_0, v_3\}\}$).[2] As it happens, each of these independent sets of propositions can also serve as a basis for $\mathscr{P}(\{v_0, v_1, v_2, v_3\})$: any subset of $\{v_0, v_1, v_2, v_3\}$ can be represented by a truth-functional combination of elements of the set of formulae. There is a certain sense, then, in which the choice of independent set by which to represent $\mathscr{P}(\{v_0, v_1, v_2, v_3\})$ doesn't really matter: each independent set of formulae can serve as a basis, and consequently, each independent set can express every other independent set (that is, each independent set is a basis for the union of all the independent sets).

But this last fact raises the questions we wish to investigate. All of those just-listed facts result from the happy assumption that $\mathscr{L}$ is materially adequate, and so that the set $A_{\mathscr{L}} = \{p, q\}$ (that is, $\{\{v_0, v_1\}, \{v_0, v_2\}\}$) is a basis for $\mathscr{P}(\{v_0, v_1, v_2, v_3\})$. However, if we look at other languages, the happy fact of material adequacy, or some generalization to infinitary truth-functions, may not obtain. As such, not every language will have a basis. Moreover, speaking imprecisely, there may be something representing independent sets of formulae (potential bases) that are, as it were, inaccessible to each other, so that some potential bases cannot express the union of the set of potential bases.

In the remainder of this paper, we will investigate these questions. Our first task will be to speak more precisely about our questions. There we will define the notion of *propositional adequacy* for languages in general. Then we will carry on our investigation of the conditions for propositional adequacy for sentential languages. We will show that for languages with infinitely many atomic sentences, either the number of truth-functional connectives or their arity must be quite large for those languages to have bases for the powersets of their valuations, or even for the union of the set of potential bases.

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[2] This observation comes to me by way of Warren Goldfarb, whose conversation was the impetus for this paper. I must, however, selfishly claim all errors as my own.

2 Defining propositional adequacy

First of all, we have spoken loosely about formulae "representing" sets of interpretations. So let us make this more precise with the following: Let $\mathscr{R}_{\mathscr{L}}$ be the set of interpretations of $\mathscr{L}$.

Definition 1. *Let φ be a formula of a language $\mathscr{L}$, and S a set of interpretations of $\mathscr{L}$. Then φ* represents *S if and only if for every interpretation v of $\mathscr{L}$, $v(\varphi) = T$ if and only if $v \in S$. If φ represents S, then we will write S as S_φ. If X is a set of formulae, then we will let $G_X = \{S_\varphi \subseteq \mathscr{R}_{\mathscr{L}} \mid \varphi \in X\}$. We will say that X represents G_X just if for each member of G_X, it's representative in X is unique.*[3]

While our main concern will be the adequacy of a language (and so the existence of formulae with certain semantic properties), our proofs will be largely semantic in nature, leaving the existence of syntactic objects for the last step. In the above, we waved our hands at this by talking of "potential bases." But in our example above, it was easy to talk about potential bases, since every potential basis was a real one. Thus we were able to say that $\{p, q\}, \{p, \varphi_7\}$ and their independent ilk are bases for $\mathscr{P}(\{v_0, v_1, v_2, v_3\})$, since, *because of the material adequacy of $\mathscr{L}$*, every subset of $\{v_0, v_1, v_2, v_3\}$ has a *formula* representing it. This is terribly trivial, of course, until we try to say what it is for a language to have *potential* bases. It would seem that the notion of a basis is ill-conceived for being founded on trivialities.

But we can get away from our dependence on material adequacy by considering functions on sets of interpretations of a language, functions that correspond to truth-functional connectives, as follows: For $\mathscr{L}$ a formal language, let $\mathscr{R}_{\mathscr{L}}$ be the set of interpretations of $\mathscr{L}$. For α an ordinal,

Definition 2. *Let $\star$ be an α-placed truth-functional connective of a language $\mathscr{L}$, and let $\widetilde{\star} : {}^\alpha\{T, F\} \to \{T, F\}$ be the α-placed truth-function defining it. Then the function $\widehat{\star} : {}^\alpha\mathscr{P}(\mathscr{R}_{\mathscr{L}}) \to \mathscr{P}(\mathscr{R}_{\mathscr{L}})$ is defined as follows:*

For $s \in {}^\alpha\mathscr{P}(\mathscr{R}_{\mathscr{L}})$, $v \in \mathscr{R}_{\mathscr{L}}$, let s_v be the binary α-sequence such that for $\beta < \alpha$, $s_v(\beta) = T$ if and only if $v \in s(\beta)$. Then

$$\widehat{\star}(s) = \{v \in \mathscr{R}_{\mathscr{L}} \mid \widetilde{\star}(s_v) = T\}.$$

For convenience, we will say that $\widehat{\star}$ is the propositional connective *associated with the truth-functional connective $\star$, and with the truth-function $\widetilde{\star}$.*

[3]This last definition is simply to avoid some needless complications.

It is easy to see that if formulae φ_β represent sets of interpretations S_β, then $\star(\langle\varphi_\beta\rangle_{\beta<\alpha})$ represents $\widehat{\star}(\langle S_\beta\rangle_{\beta<\alpha})$.

This allows us to talk of independent sets of propositions, in the following sense.

Definition 3. *Let $G \subseteq \mathscr{P}(\mathscr{R}_{\mathscr{L}})$ be a set of propositions. We will say G is* independent *if and only if for every $S \subseteq G$ there is a unique*[4] *interpretation v_S^G of $\mathscr{L}$ such that for any $E \in G$, $v_S^G \in E$ if and only if $E \in S$.*

It is obvious (though the details may be tedious) that if a set X of formulae (with no two formulae equivalent) are such that for each subset there is a unique valuation v_X making just the members of that subset true, then G_X is and independent set of propositions. It follows that if $\mathscr{L}$ is materially adequate then every independent set of propositions has a set of representatives.

From here it is a short journey to define propositional adequacy, our topic.

Let $\dagger$ be a set of truth-functional connectives. Unless we specify otherwise, in the context of an assumed language $\mathscr{L}$, $\dagger$ will be the set of connectives of $\mathscr{L}$.

Definition 4. *For $G \subseteq \mathscr{P}(\mathscr{R}_{\mathscr{L}})$ and $\star \in \dagger$ an α-placed truth-functional connective, define $G^\star = \{\widehat{\star}(s) \mid s \in {}^\alpha G\}$.*

We will be interested in using the correlates of several truth-functional connectives to generate propositions from given sets of sentences.

Definition 5. *Let $\dagger$ be a set of truth-functional connectives, $G \subseteq \mathscr{P}(\mathscr{R}_{\mathscr{L}})$. We define $G^\dagger$ as follows:*

$$\begin{aligned} G_0^\dagger &= G \\ G_{n+1}^\dagger &= \textstyle\bigcup_{\star\in\dagger}(G_n^\dagger)^\star \cup G_n^\dagger \\ G^\dagger &= \textstyle\bigcup_{\beta<\omega} G_\beta^\dagger \end{aligned}$$

So $G^\dagger$ is the set of propositions generated from G using the correlates of truth-functional connectives in $\dagger$.

[4]The uniqueness condition can be omitted, though this leads to less interesting results. Many of the cardinality results of Section 3 still obtain, though the preliminary results need to be modified slightly. Similarly, the general results about commensurability in Section 4 obtain as well. However, the incommensurability results in Section 24 are mostly trivial if the uniqueness condition is dropped.

Definition 6. *We will say $G \subseteq \mathscr{P}(\mathscr{R}_{\mathscr{L}})$ is a* potential basis *for $\mathscr{L}$ just if $G^{\dagger} = \mathscr{P}(\mathscr{R}_{\mathscr{L}})$. A set for formulae X is a* basis *for $\mathscr{L}$ just if it represents G_X and G_X is a potential basis.*

Definition 7. *A language is* propositionally adequate *just if it has a basis.*

We will also concern ourselves with a weaker condition. Let $IND[\mathscr{L}] = \{G \subseteq \mathscr{P}(\mathscr{R}_{\mathscr{L}}) \mid G \text{isindependent}\}$. Then

Definition 8. *A set $G \subseteq \mathscr{P}(\mathscr{R}_{\mathscr{L}})$ is a* potential λ-semantic basis *for $\mathscr{L}$ just if $G^{\dagger} = \bigcup IND[\mathscr{L}]$ and $|G| = \lambda$ (for λ a cardinal). A set of formulae X is a* λ-semantic basis *for $\mathscr{L}$ just if X represents G_X and G_X is a potential λ-semantic basis.*

Definition 9. *A language is* λ-semantically adequate *if it has a λ-semantic basis.*

Definition 10. *A language $\mathscr{L}$ is* semantically adequate *if it is λ-semantically adequate for all $\lambda \in W(|A_{\mathscr{L}}|)$.*

We should note, given these definitions, that propositional adequacy is a very close relative of material adequacy. In fact, for sentential languages, which shall be our present concern, propositional adequacy is a generalization of material adequacy to the case of infinitary truth-functions. However, we will do well to note that this is not so with propositional adequacy conceived in general, for the definitions given above can be applied just as well to first-order languages. For first-order language, the interpretations of concern are not valuations of atomic sentences, but models of the language—mathematical objects distinct from the valuations of sentential languages into which truth-functions are built.

In the remainder of this paper we will discuss the conditions of propositional and semantic adequacy for sentential languages. Most of our results, which are negative and limitative, will be about semantic adequacy, since our result for propositional adequacy is easily proved (see Theorem 22). Moreover, it is obvious that:

Proposition 11. *If a language is propositionally adequate then it is semantically adequate.*

Before moving on, we should note that the definitions given for propositional and semantic adequacy and their relata concern not just sentential logic, which will concern us in this paper, but also first-order languages. The results presented below for sentential logic do not translate directly to the first-order case, since the latter face the added wrinkle of quantifiers. These open the possibility of interpretations of the language for which there are objects in the domain with no expression in the language representing them. This also brings in attendant worries from the Löwenheim-Skolem Theorem.

3 Counting independent sets

Our main results in this paper will rely on cardinality arguments, so we shall need to discern the number of independent sets. If there are too many independent sets for a language to express, then the language is neither semantically nor propositionally adequate.

We noted above that, even in the simple language in the introduction, there is more than one independent set. Let us focus a bit more on this fact. Consider the truth table for the following formulae of the language discussed in the introduction.

p	q	φ_7
T	T	T
T	F	F
F	T	F
F	F	T

.

As noted earlier, the sets represented by $\{p, q\}$ and $\{p, \varphi_7\}$ are both independent. One way of seeing this is by noting that each row of the truth-table represents a valuation of the atomic formulae of the language (that is, p and q). Now note that the columns of p and φ_7 end up permuting the rows of the truth-table of p and q:

p	q		p	φ_7
T	T	$\rightarrow$	T	T
T	F	$\rightarrow$	T	F
F	T	$\searrow$	F	F
F	F	$\nearrow$	F	T

This permutation of rows can equally be seen as permutation of valuations, since, looking at a truth-table where the left-most columns are the atomic propositions of a language, each row represents a valuation. So if we look at the sets represented by each of these formulae, we can see a similar permutation phenomenon:

$$\begin{array}{ccccc} p & q & & p & \phi_7 \\ \left\{\begin{array}{c} v_0 \\ v_1 \\ \\ \\ \end{array}\right\} & \left\{\begin{array}{c} v_0 \\ \\ v_2 \\ \\ \end{array}\right\} & \begin{array}{c} \rightarrow \\ \rightarrow \\ \times \\ \\ \end{array} & \left\{\begin{array}{c} v_0 \\ v_1 \\ \\ \\ \end{array}\right\} & \left\{\begin{array}{c} v_0 \\ \\ \\ v_3 \end{array}\right\} \end{array}$$

Here, starting with the independent set $\{\{v_0, v_1\}, \{v_0, v_2\}\}$, we may apply the permutation

$$b(x) = \begin{cases} v_3 & x = v_2 \\ v_2 & x = v_3 \\ x & \text{otherwise} \end{cases}$$

on valuations to obtain the independent set $\{\{b(v_0), b(v_1)\}, \{b(v_0), b(v_2)\}\} = \{\{v_0, v_1\}, \{v_0, v_3\}\}$.

In the rest of this section, we will show that all the independent sets of a language can be obtained through such permutations of valuations within a given independent set. But a given independent set is always easy to come by, for it is the set of propositions represented by the atomic sentences of the language. As such, the number of independent sets of propositions of a language is derived from the number of permutations of the valuations of a language. I say "derived" because in permuting the valuations, some permutations will turn up the same set (think: some permutations of rows will generate the same columns, only in a different order). So we will need to modulate for these multiple hits.

Before showing this, though, we will need to deal with a slight complication that may appear in the cases of languages with infinitely many valuations. In the case of languages with only 2^n many valuations where n is finite, all independent sets are of cardinality n. In the infinite case, however, this is different. The set of valuations of a language with infinitely many atomic sentences is equinumerous with the power-set of those atomic sentences. Whether or not there are independent sets not equinumerous with the set of atomic sentences depends on whether or not GCH is true, for if it is true, then the only sets whose powersets are equinumerous with the powerset of atomic sentences are those equinumerous with the set of atomic sentences. If GCH is false, however, there might be sets of greater or lesser size than the set of atomic sentences, with powersets equinumerous to the powerset of atomic sentences. In this case, we would have at least one independent set (the set represented by the atomic sentences) of size κ, and another independent set of different cardinality: if $|G| \neq \kappa$ but $|\mathscr{P}(G)| = |\mathscr{P}(\kappa)|$, then there can be a unique valuation for each subset of G satisfying the definition for independence.

On to our results. For any set K let $B(K)$ be the set of bijections from K to itself. Moreover, for $b \in B(K)$, $Y \in \mathscr{P}(K)$, we let $b(Y) = \{b(v) \mid v \in Y\}$. Additionally, for $X \subseteq \mathscr{P}(K)$, we let $b(X) = \{b(Y) \mid Y \in X\}$.

In the following, let $\mathscr{L}$ be a language with κ atomic sentences, assume $\lambda \in W(\kappa)$.

Proposition 12. *Let* $G \in IND_\lambda[\mathscr{L}]$, $b \in B(\mathscr{R}_\mathscr{L})$. *Then* $b(G) \in IND_\lambda[\mathscr{L}]$.

Proof. Let G be an independent set of size λ, $b \in B(\mathscr{R}_\mathscr{L})$. We first show $b(G)$ independent. For $S \subseteq B(G)$, each member of S is equal to $b(E)$ for some proposition $E \in G$. Let $S' = \{E \in G \mid b(E) \in S\}$, and $v_{S'}$ be the unique valuation such that, for $E \in G$, $v_{S'} \in E$ if and only if $E \in S'$. Now note: for $b(E) \in b(G)$,

$$\begin{aligned} b(v_{S'}) \in b(E) \quad &\text{if and only if} \quad v_{S'} \in E \\ &\text{if and only if} \quad E \in S' \\ &\text{if and only if} \quad b(E) \in S. \end{aligned}$$

For uniqueness of $v_{S'}$, note that if $b(v')$ is such that for any $b(E) \in b(G)$, $b(v') \in b(E)$ if and only if $b(E) \in S$, then $v' \in E$ if and only if $E \in S'$; thus $v_{S'} = v'$, and so $b(v_{S'})$ is unique.

Regarding the cardinality of $b(G)$, note that $b(E) = b(F)$ if and only if $E = F$. If $b(E) \neq b(F)$, without loss of generality assume $b(v) \in b(E) - b(F)$. Then $v \in E - F$. Conversely, again without loss of generality, if $v \in E - F$, then $b(v) \in b(E) - b(F)$. Consequently, $|b(G)| = |G|$. □

Proposition 13. *(AC) If* $IND_\lambda[\mathscr{L}] \neq \emptyset$, *then* $|B(\lambda)| \times |IND_\lambda[\mathscr{L}]| = |B(2^\kappa)|$.

Proof. Let $C \in IND_\lambda[\mathscr{L}]$. By Proposition 12, for each $b \in B(\mathscr{R}_\mathscr{L})$, $b(C) \in IND[\mathscr{L}]$. We need only to discount cases where two or more permutations generate the same independent set.

Note, however, that to each $b \in B(\mathscr{R}_\mathscr{L})$ there is a bijection $h_b : C \to b(C)$ such that for $E \in C$, $h_b(C) = b(C)$. Thus, for $H \in IND[\mathscr{L}]$, the number of permutations b such that $b(C) = H$ is the number of bijections from C to H. Since $|G| = |H| = \lambda$, that number of bijections is $|B(\lambda)|$.

Thus, $|B(\lambda)| \times |IND_\lambda[\mathscr{L}]| = |B(\mathscr{R}_\mathscr{L})| = |B(2^\kappa)|$. □

Thus, to discern the cardinality of $IND_\lambda[\mathscr{L}]$, for any $\lambda \in W(\kappa)$, we need only discover the cardinality of the correlated $B(\lambda)$ and $B(2^\kappa)$. Furthermore,

Corollary 14. *(AC)* $|B(\kappa)| \times |IND[\mathscr{L}]| = |B(2^\kappa)| \times |W(\kappa)|$.

Proof. If $W(\kappa) = \{\kappa\}$, the result is immediate. If $|W(\kappa)| > 1$, then κ is infinite, so for each $\lambda \in W(\kappa)$, $|B(\kappa)| = 2^\kappa = 2^\lambda = |B(\lambda)|$. □

This result immediately gives us the correct answer for how many independent sets there are for languages with finitely many atomic sentences. Let us first note that in our initial example (see the introduction) of a language with two atomic sentences, there are twelve pairs of independent propositions, namely the (unordered) pairs of the set containing the propositions represented (assuming the usual stock or connectives) by "p", "$\neg p$" (φ_{10}), "q", "$\neg q$" (φ_{13}), "$p \equiv q$" (φ_7), and "$\neg p \equiv q$" (φ_{14}), where no pair includes a sentence and its negation.

More generally:

Proposition 15. *Let $\mathscr{L}$ be a language with $n < \omega$ atomic sentences.*[5] *Then*

$$|IND[\mathscr{L}]| = \frac{2^n!}{n!}$$

Proof. Since, for any finite k, the number of bijections from k to k is $k!$, and $|W(k)| = 1$, Corollary 14 yields the result. □

Much more interesting is the case where κ is an infinite cardinal. The case of independent sets for languages with $\kappa \geq \omega$ many atomic sentences might seem different because, if GCH fails, $W(\kappa)$ may be non-empty. However, just as in the finite case, the size of each $IND_\lambda[\mathscr{L}]$, and thus of $IND[\mathscr{L}]$, depends only on κ.

Corollary 16. *(AC) Let κ be an infinite cardinal, then*

$$|IND[\mathscr{L}]| = 2^{2^\kappa}.$$

[5]Though Corollary 14 relies on the Axiom of Choice, a finite version of it can be proved without the axiom. Hence this proposition does not require AC for its proof.

Proof. $|W(\kappa)| \leq 2^\kappa$, so the right-hand side of Corollary 14 is simply $|B(2^\kappa)| = 2^{2^\kappa}$. The left-hand side is $2^\kappa \times |IND[\mathscr{L}]|$, so if $|IND[\mathscr{L}]| < 2^{2^\kappa}$, $2^\kappa \times |IND[\mathscr{L}]| < 2^{2^\kappa}$ as well. $|IND[\mathscr{L}]| \leq 2^{2^\kappa}$ because $IND[\mathscr{L}] \subseteq \mathscr{P}(\mathscr{P}(\mathscr{R}_{\mathscr{L}}))$, so $|IND[\mathscr{L}]| = 2^{2^\kappa}$. □

4 Commensurability

The previous section showed that there are many different sets of independent propositions. Let us return for a moment to the case examined in the introduction. In section 3 we saw that this language, with exactly two atomic sentences, has twelve semantically independent sets. The language presented in the introduction was assumed to have a materially adequate set of truth-functional connectives. As a consequence, the language is propositionally adequate, and so also semantically adequate.

This section examines some necessary conditions for the semantically adequacy of a language, where the language has infinitely many atomic propositions. To do this, we introduce the notion of *commensurability*. The idea behind commensurability is this: suppose one were to be given, not necessarily the propositions represented by the atomic sentences of a language, but any arbitrary independent set of propositions. We have seen that for such languages, there are as many independent sets as there are propositions of that language. Can this arbitrarily chosen independent set express the propositions of another independent set using the propositional correlates of the truth-functional connectives of $\mathscr{L}$? If so, roughly speaking, the two sets are commensurable.

The goal of this section is to prove that, for a language with infinitely many atomic sentences, if the language is semantically adequate, then its truth-functional connectives must be relatively large (in a sense to be defined) in either number or in the arity of their supremum. We will do this by showing that if a language has truth-functional connectives not relatively large in either number or arity of their supremum, then there are independent sets that are not commensurable.

Let us begin by making the notion of commensurability more precise.

Definition 17. *Let G and H be two independent sets of propositions of a language $\mathscr{L}$, and let † be a set of truth-functions. Then we say H is †-*commensurable *with G if and only if $H \subseteq G^\dagger$. We write this $G \overset{\dagger}{\looparrowright} H$. H is*

†-incommensurable *with* G *just if* $G \overset{\dagger}{\not\rightarrow} H$.

We can discern some basic conditions for and facts about commensurability, as indicated in the following. We will focus on the infinite cases. Hereafter, the languages under consideration will all have infinitely many atomic sentences, and every independent set we consider will have infinite cardinality.

Proposition 18. *(AC) Let † be a set of truth-functional connectives, where the arity of each* $\star \in \dagger$ *is given by* $\alpha_\star$*. Then* $|G^\dagger| \leq |\dagger| \cdot |G|^{|\sup_{\star\in\dagger}\alpha_\star|^{(\omega)}}$.

Proof. We show by induction that for each $n \in \omega$, $|G^\dagger_n| \leq |\dagger| \cdot |G|^{|\sup_{\star\in\dagger}\alpha_\star|^n}$. The result follows since

$$|G^\dagger| \leq \sum_{n\in\omega} |G^\dagger_n| \leq |\omega| \cdot |\dagger| \cdot |G|^{|\sup_{\star\in\dagger}\alpha_\star|^{(\omega)}} = |\dagger| \cdot |G|^{|\sup_{\star\in\dagger}\alpha_\star|^{(\omega)}}.$$

The base case is immediate; suppose the hypothesis holds for n. Then since $|G^\dagger_{n+1}| \leq \sum_{\star\in\dagger} |(G^\dagger_n)^\star|$ and $|(G^\dagger_n)^\star| \leq |{}^{\alpha_\star}G^\dagger_n|$,

$$|G^\dagger| \leq \sum_{\star\in\dagger} |G^\dagger_n|^{|\alpha_\star|} \leq |\dagger| \cdot (|G|^{|\sup_{\star\in\dagger}\alpha_\star|^n})^{|\sup_{\star\in\dagger}\alpha_\star|} = |\dagger| \cdot |G|^{|\sup_{\star\in\dagger}\alpha_\star|^{n+1}}.$$

□

Corollary 19. *For* $|\sup_{\star\in\dagger}\alpha_\star|$ *infinite,* $|G^\dagger| \leq |\dagger| \cdot |G|^{|\sup_{\star\in\dagger}\alpha_\star|}$.

Proposition 20. *(AC) Let* κ *be infinite,* $\lambda \in W(\kappa)$*. Then*

$$|\bigcup IND_\lambda[\mathscr{L}]| = 2^{2^\kappa}.$$

Proof. Let

$$U_\kappa = \{E \in \mathscr{P}(\mathscr{R}_\mathscr{L}) \mid |E| = |\mathscr{R}_\mathscr{L} - E|\}.$$

Order $\mathscr{R}_\mathscr{L}$ by 2^κ, and note that the set $LIM(2^\kappa)$ of limit ordinals in 2^κ, and its complement, are both of cardinality 2^κ. As in [2], 48-9, there are 2^{2^κ} subsets of $LIM(2^\kappa)$ of size 2^κ, and the complement of each of these is also of size 2^κ. By the ordering of $\mathscr{R}_\mathscr{L}$ by 2^κ, to each subset of 2^κ there is a corresponding subset; thus $|U_\kappa| = 2^{2^\kappa}$. We show that $\bigcup IND_\lambda[\mathscr{L}]| = U_\kappa$.

Let $G \in IND_\lambda[\mathscr{L}]$ and $E \in G$. Note that $|\{S \in \mathscr{P}(G)) \mid E \in S\}| = |\mathscr{P}(G)| = 2^\lambda = 2^\kappa$. Since G is independent, if S is a member of $\{S \in$

$\mathscr{P}(G)) \mid E \in S\}$, then for any $E' \in G$, $v_S^G \in E'$ iff $E \in S$. Because $S \in \{S \in \mathscr{P}(G)) \mid E \in S\}$, $E \in S$; consequently $v_S^G \in E$ for all $S \in \{S \in \mathscr{P}(G)) \mid E \in S\}$. Thus $|E| = 2^\kappa$. A similar argument shows $|\mathscr{R}_{\mathscr{L}} - E| = 2^\kappa$. So $\bigcup IND_\lambda[\mathscr{L}] \subseteq U\kappa$.

Let $u \in U_\kappa$, and pick an arbitrary $E \in \bigcup IND_\lambda[\mathscr{L}]$. By the above, $E \in U_\kappa$. Since $|E| = |u|$ and $|\mathscr{R}_{\mathscr{L}} - E| = |\mathscr{R}_{\mathscr{L}} - u|$, let b be bijection between E and u, and b' a bijection between their complements. The union b'' of b and b' is a bijection as well, so $b''(E) = u$. There is a $G \in IND_\lambda[\mathscr{L}]$ with $E \in G$, and as we showed in Proposition 12, $b''(G) \in IND_\lambda[\mathscr{L}]$, so since $u \in b''(G)$, $u \in \bigcup IND_\lambda[\mathscr{L}]$.

□

We can now address our aims for this section more directly. We will say that an ordinal α such that $2^\alpha < 2^{2^\kappa}$ is *reasonably small.*

Proposition 21. *(AC) Let κ be infinite, and let † be a set of less than 2^{2^κ} truth-functional connectives, such that $\sup_{\star\in\dagger} \alpha_\star$ is reasonably small. Then for $G \in IND_\lambda[\mathscr{L}]$, $|G^{|\sup_{\star\in\dagger} \alpha_\star|}| = \lambda^{|\sup_{\star\in\dagger} \alpha_\star|} < 2^{2^\kappa}$.*

Proof. If $\sup_{\star\in\dagger} \alpha_\star \geq \lambda$, then $\lambda^{|\sup_{\star\in\dagger} \alpha_\star|} = 2^{|\sup_{\star\in\dagger} \alpha_\star|} < 2^{2^\kappa}$. Otherwise, $\lambda^{|\sup_{\star\in\dagger} \alpha_\star|} = 2^{|\sup_{\star\in\dagger} \alpha_\star|} < 2^{2^\kappa}$ unless $2^{|\sup_{\star\in\dagger} \alpha_\star|} < \lambda$, in which case: if $2^{|\sup_{\star\in\dagger} \alpha_\star|}$ is finite, then so is $|\sup_{\star\in\dagger} \alpha_\star|$, so $\lambda^{2^{|\sup_{\star\in\dagger} \alpha_\star|}} = \lambda < 2^{2^\kappa}$. If $2^{|\sup_{\star\in\dagger} \alpha_\star|}$ is infinite, then $\lambda^{|\sup_{\star\in\dagger} \alpha_\star|} \leq 2^\kappa < 2^{2^\kappa}$. (See [2, 51] for details of these cardinality arguments.) □

Theorem 22. *(AC) Let κ be infinite, and let † be a set of less than 2^{2^κ} truth-functional connectives, such that $\sup_{\star\in\dagger} \alpha_\star$ is reasonably small. Then for all $G \in IND_\lambda[\kappa]$, $\bigcup IND_\lambda[\kappa] \nsubseteq G^\dagger$.*

Proof. By Proposition 21, $|G^{|\sup_{\star\in\dagger} \alpha_\star|}| = \lambda^{|\sup_{\star\in\dagger} \alpha_\star|} < 2^{2^\kappa}$.

By Corllary 19 and Proposition 20,

$$|G^\dagger| \leq |\dagger| \cdot |G|^{|\sup_{\star\in\dagger} \alpha_\star|} \leq |\dagger| \cdot \lambda^{|sup_{\star\in\dagger}\alpha_\star|} < 2^{2^\kappa} = |\bigcup IND_\lambda[\mathscr{L}]|.$$

□

Corollary 23. *(AC) Let κ be infinite, and let $\mathscr{L}$ be a language with truth-functional connectives †, which has size less than 2^{2^κ} and such that $\sup_{\star\in\dagger} \alpha_\star$ is of reasonably small arity. Then $\mathscr{L}$ is not propositionally adequate.*

Proof. This easily follows since $\bigcup IND_\lambda[\mathscr{L}] \subseteq \mathscr{P}(\mathscr{R}_\mathscr{L})$.[6] □

So we have discovered a necessary condition for both semantic and propositional adequacy. There is a further result regarding the former: Let

$$IND_\lambda[\mathscr{L}]^\dagger = \{G^\dagger \mid G \in IND_\lambda[\mathscr{L}]\}.$$

Theorem 24. *Let κ be an infinite cardinal, $\mathscr{L}$ a language with $\dagger$ its set of truth-functional connectives, where $|\dagger| < 2^{2^\kappa}$ and $|\sup_{\star\in\dagger}\alpha_\star|$ reasonably small. Then if $IND_\lambda[\mathscr{L}] \neq \emptyset$, there are 2^{2^κ} pairs of sets $G, H \in IND_\lambda[\mathscr{L}]$ such that $G \not\subseteq H^\dagger$ and $H \not\subseteq G^\dagger$. (That is, there are 2^{2^κ} incommensurable pairs of independent sets of propositions of size λ.)*

Proof. The theorem follows from two facts. One is that $|IND_\lambda[\mathscr{L}]^\dagger| = 2^{2^\kappa}$. The second is a general fact about linear orderings: for X a set of cardinality δ, linearly ordered by $\sqsubset$, and $\gamma < \delta$ a cardinal. Then there is an $x \in X$ such that $|\{y \in X \mid y < x\}| \geq \gamma$.

To see that these two facts suffice, consider that if there is no pair of incommensurable independent sets of size λ, then $\subset$ linearly orders $IND_\lambda[\kappa]^\dagger$. This induces a partial ordering $\sqsubset$ of $IND_\lambda[\mathscr{L}]$ whereby $G \sqsubset H$ just if $G^\dagger \subset H^\dagger$. By the axiom of choice, for each $J \in IND_\lambda[\mathscr{L}]^\dagger$ pick exactly one $G_J \in IND_\lambda[\mathscr{L}]$ such that $J = G_J^\dagger$, it follows that $\sqsubset$ linearly orders $\{G_J \mid J \in IND_\lambda[\mathscr{L}]^\dagger\}$. Thus, these two facts yield that there is a $G_0 \in IND_\lambda[\mathscr{L}]$ such that

$$|\{H \in IND_\lambda[\mathscr{L}]^\dagger \mid H \subset G_0\}| \geq 2^\kappa.$$

But this contradicts the fact that $|G_0| = \lambda < 2^\kappa$. Since there is one incommensurable pair G and H, if b is a bijection of $\mathscr{R}_\mathscr{L}$ to itself, $b(G)$ and $b(H)$ will constitute another pair, and there will be 2^{2^κ} such bijections generating new pairs.

The general fact about linear orderings is proved as follows: If not, then for f a bijection between δ and X, and

$$A = \{\eta < \delta \mid \forall \rho < \eta(f(\rho) \sqsubset f(\eta))\},$$

$\sup(A)$ is not greater than γ, and has no greatest element. Note that, for each $\alpha, \beta \in A$ with $\alpha < \beta$, $|\{f(\varsigma) \in X \mid f(\alpha) \sqsubset f(\varsigma) \sqsubset f(\beta)\}| < \gamma$. But then

[6] In fact, it follows directly from Corollary 19. A result similar to this is discussed, and sketched, by G.E.M. Anscombe in [1, 134-7].

$|X| \leq \gamma \times |(A \times A)| = \gamma \times |A| < \delta$, contradicting our assumption about the size of X.

Finally, let us show that $|IND_\lambda[\mathscr{L}]^\dagger| = 2^{2^\kappa}$. We shall first note that $|\bigcup IND_\lambda[\mathscr{L}]^\dagger| = 2^{2^\kappa}$, by Proposition 20 and since

$$\bigcup IND_\lambda[\mathscr{L}] \subseteq \bigcup IND_\lambda[\mathscr{L}]^\dagger \subseteq \mathscr{P}(\mathscr{R}_\mathscr{L}),$$

and $|\mathscr{P}(\mathscr{R}_\mathscr{L})| = 2^{2^\kappa}$.

Now consider that the function $f(G) = G^\dagger$ on $|IND_\lambda[\mathscr{L}]$ is (trivially) onto $IND_\lambda[\kappa]^\dagger$, hence

$$|IND_\lambda[\mathscr{L}]^\dagger| \leq |IND_\lambda[\mathscr{L}]| = 2^{2^\kappa}$$

(by Proposition 20). Note that by Corollary 19,

$$|\bigcup IND_\lambda[\mathscr{L}]^\dagger| \leq |IND_\lambda[\mathscr{L}]^\dagger| \times \lambda^{|\sup_{\star \in \dagger} \alpha_\star|};$$

further, by Proposition 21, $\lambda^{|\sup_{\star\in\dagger}\alpha_\star|} < 2^{2^\kappa}$. Thus, if $|IND_\lambda[\mathscr{L}]^\dagger| < 2^{2^\kappa}$, then so is $|\bigcup IND_\lambda[\mathscr{L}]^\dagger|$, contradicting what was just shown. □

Before moving on, it is worth noting that the techniques used in this section prove another result, somewhat more curious: that for languages that are not semantically adequate, an unrestricted semantic correlate to the Craig Interpolation Theorem is false. Recall the theorem:

Craig Interpolation Theorem 25. *If φ and ψ are formulae of a language such that $\varphi \models \psi$, then there a formula θ, called an* interpolant*, such that $\varphi \models \theta$, $\theta \models \psi$, and θ contains only atomic formulae common to both φ and ψ.*

In concerning sentences, this is clearly a theorem about syntactic objects. It's semantic correlate would be:

Semantic Interpolation Principle 26. *If E and F are propositions of a language such that $F \subseteq E$, then there is a proposition C such that $F \subseteq C \subseteq E$ and $C \in (G \cap H)^\dagger$, for any independent sets G, H with $E \in G^\dagger$ and $F \in H^\dagger$.*

The Semantic Interpolation Principle is false unless it is limited by the hypotheses of Theorem 24 plus some other conditions. This is easy to explain; we will prove it in the balance of this section. The explanation is that the Craig Interpolation Theorem is true precisely because the formulae it deals with are all generated from one privileged set of formulae, representing only one independent set: the atomic sentences. As such, it needs to deal only with what can be generated from that set, and need not be concerned with expressing the many propositions of the language. The Semantic Interpolation Principle traffics *between* independent sets, and so is required to deal with the many many propositions of $\mathscr{L}$. As a consequence, it is limited by the arity of its truth-functions. We formalize this in what follows.

Say that a truth-functional connective $\star \in \dagger$ is *all-null similar* if for some $s \in {}^{\alpha_\star}G$, G an independent set, there are valuations u, v such that $v \in \widehat{\star}(s)$ if and only if $u \in \widehat{\star}(s)$ and for any $E \in G$, $v \in E$ if and only if $u \notin G$. (Think of s as a formula that ends up true on valuations that are mirror images of each other: v has true entries where and only where u has false entries.) Further, for $A, B \subseteq \mathscr{R}_{\mathscr{L}}$, say that a bijection b from $\mathscr{R}_{\mathscr{L}}$ onto itself is *A, B-slippy over valuations u and v* if $b(u) = v$, $b(v) = u$, $b(A) = A$, and $b(B) = B$.

Theorem 27. *Let κ be an infinite cardinal, $\lambda \in W(\kappa)$, $\mathscr{L}$ a language with $|A_{\mathscr{L}}| = \kappa$ and $\dagger$ its set of truth-functional connectives, where $|\dagger| < 2^{2^\kappa}$ and $|\sup_{\star\in\dagger} \alpha_\star|$ reasonably small, and has an all-null similar connective. Then if $IND_\lambda[\mathscr{L}] \neq \emptyset$, there are 2^{2^κ} sets $G, H \in IND_\lambda[\mathscr{L}]$ such that G and H are incommensurable and $(G \cap H)^\dagger \subsetneq G^\dagger \cap H^\dagger$.*[7]

The proof relies much on the machinery used to prove Theorem 24, as such we shall give only a sketch.

Proof. Let $G \in IND_\lambda[\mathscr{L}]$ and let $\star \in \dagger$ be all-null similar. Let $u, v \in \mathscr{R}_{\mathscr{L}}$ and $s \in {}^{\alpha_\star}G$ witness the all-null similarity of $\star$, and let $A = \{v \in \mathscr{R}_{\mathscr{L}} \mid v \in \widehat{\star}(s)\}$, $B = \mathscr{R}_{\mathscr{L}} - A$. The valuations u and v are either both in A or both in B, so there are 2^{2^κ} A, B-slippy bijections over u and v, so by arguments similar to those showing $|IND_\lambda[\mathscr{L}]| = 2^{2^\kappa}$, we can show that there are 2^{2^κ} independent sets H such that $H = b(G)$ for some bijection b where b is A, B-slippy over u and v. Since $\dagger$ satisfies the hypotheses of Theorem 24, there are 2^{2^κ} of these just-mentioned sets H that are incommensurable with G.

We now show that for such an H, $(G\cap H)^\dagger \subsetneq G^\dagger \cap H^\dagger$. First, $(G\cap H)^\dagger = \emptyset$: for any $E \in G$, $v \in E$ if and only if $u \notin E$ if and only if $b(u) = v \notin b(E)$. If $b(E) = E'$ for some $E' \in G$, then (without loss of generality)

[7] We can actually improve on this somewhat by allowing $\dagger$ to have the resources to express an all-null similar connective (rather than have it straightaway). However, this would require proving some lemmata that would be a bit far afield for this paper.

$E' = E - \{u\} \cup \{v\}$. But let $S' = \{F \in G \mid u \in F\}$, and pick $E'' \in S'$ distinct from E. Then consider $S = S' - \{E''\}$; since G is independent there is a there is a v_S^G such that for any $F \in G$, $v_S^G \in F$ if and only if $F \in S$. Thus, $v_S^G \in E$ but $v_S^G \notin E''$. If $v_S^G \neq u$, then $v_S^G \in E'$. But then $E' \in S'$, a contradiction since $u \notin E'$. If $v_S^G = u$, then $v_S^G \in E''$, but this too is a contradiction as $E'' \notin S$. So $G \cap H = \emptyset$, so $(G \cap H)^\dagger = \emptyset$

However, $G^\dagger \cap H^\dagger \neq \emptyset$: for any $v' \in \mathscr{R}_{\mathscr{L}}$, $v' \in \widehat{\star}(s)$ if and only if $b(v') \in \widehat{\star}(s)$, since $b(A) = A$ and $b(B) = B$. □

Corollary 28. *There are counterexamples to the Semantic Interpolation Principle if $\mathscr{L}$ satisfies the hypotheses of Theorem 27.*

Proof. We show that the Semantic Interpolation Principle implies $(G \cap H)^\dagger = G^\dagger \cap H^\dagger$ for any G, H; thus Theorem 27 provides counterexamples to it.

That $(G \cap H)^\dagger \subseteq G^\dagger \cap H^\dagger$ is trivial, and does not depend on the Semantic Interpretation Principle. So let us turn to the other inclusion: if $D \in G^\dagger \cap H^\dagger$, then think of D in two ways. On the one hand, we have D_G, D as in $G^\dagger$. We also have D_H, D as in $H^\dagger$. Thus, we have $D_G \subseteq D_H$. The Semantic Interpolation Principle gives us C such that $D_H \subseteq C \subseteq D_G$ with $C \in (G \cap H)^\dagger$. But since $D = D_H = D_G$, $C = D$ as well. Thus, $D \in (G \cap H)^\dagger$. □

Discussion of the results These results might seem idle, useless mathematics. I don't wish to claim much for them, but I do think it is worth pointing out some consequences that might be relevant to philosophy, particularly philosophy concerned with meaning, truth-conditions, and communication. Imagine we start with the speakers of a language $\mathscr{L}$ which satisfies the hypotheses of Theorem 24. Further imagine that the speakers of $\mathscr{L}$ come across speakers of another language $\mathscr{L}'$ with the same truth-functional connectives. Theorem 24 guarantees that there is an independent set of propositions G of $\mathscr{L}$ such that for each atomic sentence of $\mathscr{L}'$, the speakers of $\mathscr{L}'$ affirm that sentence under the conditions (relative to $\mathscr{L}$), specified by some unique member of G. In other words, the speakers of $\mathscr{L}'$ have as their atomic sentences what would be very complex sentences for the speakers of $\mathscr{L}$, if only the $\mathscr{L}$-speakers had the vocabulary to express them. To outside observers such as ourselves, we can see that the speakers of $\mathscr{L}$ and $\mathscr{L}'$ are in fact talking about the same world with their atomic sentences, but they can't seem to recognize this because neither has the logical vocabulary to generate sentences that represent the atomic sentences of the other speakers.

If in addition, $\mathscr{L}$ satisfies the hypotheses of Theorem 27, then it is possible that the two language groups can even utter sentences that have the same truth-conditions (as it were), but not having a means of translating their bases, might not know that they were saying the same things.

5 Remaining questions

The incommensurability results obtained above make use of the conditions under which the theorems are stated. This raises the question of where these incommensurability results *fail*: given that a language with κ atomic sentences has a set of truth-functional connectives † which has size less than 2^{2^κ}, how large must the arity of the truth-functional connectives in † be in order to eliminate the possibility of incommensurable sets?[8]

On this question, I have only attained partial results. The first is easy to see. In the remainder of the section, let $\mathscr{L}$ be a language with κ atomic sentences, κ infinite and $\lambda \in W(\kappa)$, and † the truth-functional connectives of $\mathscr{L}$, with $|\dagger| < 2^{2^\kappa}$.

Definition 29. *Let $\alpha_0(\lambda, \kappa)$ be the least ordinal α such that if $|\sup_{\star \in \dagger} \alpha_\star| = \alpha$, $\bigcup IND_\lambda[\mathscr{L}] \subseteq G^\dagger$, for any $G \in IND_\lambda[\mathscr{L}]$.*

Let us also have some notation to talk about the infinitary versions of some common truth-functional connectives.

Definition 30. *For ordinals $\alpha \leq \kappa, \beta \leq 2^\kappa$, define:*

1. *$\widehat{\wedge}_\alpha : {}^{\leq\alpha}\{T, F\} \to \{T, F\}$ such that for each $s \in {}^{\leq\alpha}\{T, F\}$, $\wedge_\alpha(s) = T$ if and only if $s(\delta)) = T$ for each $\delta \leq \beta$ on which s is defined.*

2. *$\vee_\alpha : {}^{\leq\alpha}\{T, F\} \to \{T, F\}$ such that for each $s \in {}^{\leq\alpha}\{T, F\}$, $\vee_\alpha(s) = T$ if and only if $s(\delta)) = T$ for some $\delta \leq \beta$ on which s is defined.*

3. *$N_\alpha : {}^{\leq\alpha}\{T, F\} \to \{T, F\}$ such that for each $s \in {}^{\leq\alpha}\{T, F\}$, $N_\alpha(s) = T$ if and only if $\vee_\alpha(s) = F$.*

With these definition in hand, it is easy to see the following:

Proposition 31. *(AC) Suppose GCH holds. Then $\alpha_0(\lambda, \kappa) = 2^\kappa$.*

[8] This question was posed to me by Peter Koellner, in conversation.

Proof. (Sketch) Note that if $G \in IND_\lambda[\mathscr{L}]$, each $E \in \mathscr{R}_\mathscr{L}$ has (what amounts to) a disjunctive normal form in $G^{\{\widehat{\neg},\widehat{\wedge}_\kappa,\widehat{\vee}_{2^\kappa}\}}$. By Theorem 22 and GCH, $\alpha_0(\kappa,\kappa) = 2^\kappa$. □

Proposition 31 and Theorem 22 suggest two further questions:

Question 32. *Suppose there is a* β *between* κ *and* 2^κ *with* $2^\beta = 2^{2^\kappa}$. *For* $\lambda \in W(\kappa)$ *such that* $IND_\lambda[\mathscr{L}] \neq \emptyset$*, what is* $\alpha_0(\lambda,\kappa)$*?*

That is, if there are cardinals between κ and 2^κ that are not reasonably small relative to κ, then what is $\alpha_0(\lambda,\kappa)$? The second question is a bit different:

Question 33. *Suppose all cardinals are reasonably small, but* $\sup W(\kappa) = 2^\kappa$. *Then what is* $\alpha_0(\lambda,\kappa)$*?*

In the case of this second question, we might have † such that $|\dagger| < 2^{2^\kappa}$, but with truth-functional connectives each of which has reasonably small arity, yet such that $|\sup_{\star\in\dagger}\alpha_\star| = 2^\kappa$.

At present these questions are both open. There is, however, a preliminary result that bears on both of them.

Theorem 34. *(AC) Let* $\kappa < \beta < 2^\kappa$, 2^κ *regular. Then for* $G \in IND_\lambda[\kappa]$ *and* $\dagger = \{\widehat{\neg},\widehat{\wedge}_\beta,\widehat{\vee}_\beta\}$, $\bigcup IND_\lambda[\kappa] \not\subseteq G^\dagger$.

The proof of Theorem 34 comes in three steps; the remainder of this paper is directed towards its proof. As such, in the sequel let G, κ, β, λ, and † be as in this theorem's hypothesis. Furthermore, let $\ddagger = \langle\widehat{\neg},\widehat{\wedge}_\beta\rangle$. The first step is rather hum-drum:

Proposition 35. *If* $H \subseteq \mathscr{P}(\mathscr{R}_\mathscr{L})$, $H^{\widehat{\neg}} = H_1^{\widehat{\neg}}$, $H^{\widehat{\wedge}_\beta} = H_1^{\widehat{\wedge}_\beta}$, *and* $H^{\widehat{\vee}_\beta} = H_1^{\widehat{\vee}_\beta}$.

In other words, if you start with a set of propositions and then apply (the correlate of) only one of the truth-functional connectives in †, you get nothing new after the first application. We omit the proof, which is a series of straightforward, if tedious, inductions.

The next step is to show that even introducing the rest of the connectives doesn't get us much else. That is to say,

Proposition 36. *For* $H \subseteq \mathscr{P}(\mathscr{R}_\mathscr{L})$, $H^\dagger = (((H^{\widehat{\neg}})^{\widehat{\wedge}_\beta})^{\widehat{\vee}_\beta}$.

Proof. It suffices to show that $(((H^{\widehat{\neg}})^{\widehat{\wedge}_\beta})^{\widehat{\vee}_\beta}$ is closed under $\widehat{\neg}$, $\widehat{\wedge}_\beta$, and $\widehat{\vee}_\beta$. This is shown by the following:

Distribution Law 37. *Let $A \subseteq H^{\widehat{\neg}}$. Then*

$$\widehat{\vee}_\beta(\{\widehat{\wedge}_\beta(a)\}_{a \in {}^{\leq\beta}A}) = \widehat{\wedge}_\beta(\{\widehat{\vee}_\beta(k)\}_{k \in \prod_{a \in A} ran(a)}).$$

Proof. For $v \in \mathscr{R}_{\mathscr{L}}$,

$$\begin{aligned} v \in \widehat{\vee}_\beta(\{\widehat{\wedge}_\beta(a)\}_{a \in {}^{\leq\beta}A}) \quad & \text{iff} \quad v \in \widehat{\wedge}_\beta(a) \text{ for some } a_0 \in {}^{\leq\beta}A \\ & \text{iff} \quad v \in x \text{ for every } x \in ran(a_0) \\ & \text{iff} \quad v \in \widehat{\vee}_\beta(k) \text{ for every } k \in \textstyle\prod_{a \in A} ran(a) \\ & \text{iff} \quad v \in \widehat{\wedge}_\beta(\{\widehat{\vee}_\beta(k)\}_{k \in \prod_{a \in A} ran(a)}). \end{aligned}$$

□

Thus, $(((H^{\widehat{\neg}})^{\widehat{\wedge}_\beta})^{\widehat{\vee}_\beta} = ((H^{\widehat{\neg}})^{\widehat{\vee}_\beta})^{\widehat{\wedge}_\beta}$.

Next we need the correlate of DeMorgan's Laws:

DeMorgan's Laws 38. *Let $A \subseteq \mathscr{P}(\mathscr{R}_{\mathscr{L}})$, $a \in {}^{\leq\beta}A$. Then*

$$\widehat{\neg}(\widehat{\vee}_\beta)(a) = \widehat{\wedge}_\beta(\langle \widehat{\neg} a(\eta) \rangle_{\eta \in dom(a)}).$$

$$\widehat{\neg}(\widehat{\wedge}_\beta)(a) = \widehat{\vee}_\beta(\langle \widehat{\neg} a(\eta) \rangle_{\eta \in dom(a)}).$$

Proof. Note that

$$\begin{aligned} v \in \widehat{\neg}(\widehat{\wedge}_\beta)(a) \quad & \text{iff} \quad v \notin (\widehat{\wedge}_\beta)(a) \\ & \text{iff} \quad v \in x \text{ for some } x \in ran(a_0) \\ & \text{iff} \quad v \notin a(\eta) \text{ for some } \eta \in dom(a) \\ & \text{iff} \quad v \in \widehat{\neg} a(\eta) \text{ for some } \eta \in dom(a) \\ & \text{iff} \quad \widehat{\wedge}_\beta(\langle \widehat{\neg} a(\eta) \rangle_{\eta \in dom(a)}). \end{aligned}$$

A similar argument establishes that

$$\widehat{\neg}(\widehat{\wedge}_\beta)(a) = \widehat{\vee}_\beta(\langle \widehat{\neg} a(\eta) \rangle_{\eta \in dom(a)}).$$

□

Now, to see that $(((H^{\widehat{\neg}})^{\widehat{\wedge}_\beta})^{\widehat{\vee}_\beta}$ is closed under $\widehat{\neg}$, $\widehat{\wedge}_\beta$, and $\widehat{\vee}_\beta$, let $s \in {}^{\leq\beta}(((H^{\widehat{\neg}})^{\widehat{\wedge}_\beta})^{\widehat{\vee}_\beta}$ and $E \in (((H^{\widehat{\neg}})^{\widehat{\wedge}_\beta})^{\widehat{\vee}_\beta}$. Then,

$$\begin{array}{rcll} \widehat{\neg}(E) \in ((((H^{\widehat{\neg}})^{\widehat{\wedge}_\beta})^{\widehat{\vee}_\beta})^{\widehat{\neg}} & = & ((H^{\widehat{\neg}})^{\widehat{\vee}_\beta})^{\widehat{\wedge}_\beta} & \text{By DeMorgan's Laws} \\ & = & ((H^{\widehat{\neg}})^{\widehat{\vee}_\beta})^{\widehat{\wedge}_\beta} & \text{As } ((H^{\widehat{\neg}})^{\widehat{\wedge}_\beta})^{\widehat{\vee}_\beta} = ((H^{\widehat{\neg}})^{\widehat{\vee}_\beta})^{\widehat{\wedge}_\beta} \\ \\ \widehat{\vee}_\beta(s) \in (((H^{\widehat{\neg}})^{\widehat{\wedge}_\beta})^{\widehat{\vee}_\beta})^{\widehat{\vee}_\beta} & = & (((H^{\widehat{\neg}})^{\widehat{\wedge}_\beta})^{\widehat{\vee}_\beta}) & \text{By Proposition 35} \\ \\ \widehat{\wedge}_\beta(s) \in (((H^{\widehat{\neg}})^{\widehat{\wedge}_\beta})^{\widehat{\vee}_\beta})^{\widehat{\wedge}_\beta} & = & (((H^{\widehat{\neg}})^{\widehat{\vee}_\beta})^{\widehat{\wedge}_\beta})^{\widehat{\wedge}_\beta} & \text{As } ((H^{\widehat{\neg}})^{\widehat{\wedge}_\beta})^{\widehat{\vee}_\beta} = ((H^{\widehat{\neg}})^{\widehat{\vee}_\beta})^{\widehat{\wedge}_\beta} \\ & = & ((H^{\widehat{\neg}})^{\widehat{\vee}_\beta})^{\widehat{\wedge}_\beta} & \text{By Proposition 35} \\ & = & ((H^{\widehat{\neg}})^{\widehat{\wedge}_\beta})^{\widehat{\vee}_\beta} & \text{As } ((H^{\widehat{\neg}})^{\widehat{\wedge}_\beta})^{\widehat{\vee}_\beta} = ((H^{\widehat{\neg}})^{\widehat{\vee}_\beta})^{\widehat{\wedge}_\beta}. \end{array}$$

□

The final step of the proof of Theorem 34 is the following:

Proposition 39. *(AC) If 2^κ is regular, then $\bigcup IND_\lambda[\kappa] - ((G^{\widehat{\neg}})^{\widehat{\wedge}_\beta})^{\widehat{\vee}_\beta} \neq \emptyset$.*

Proof. Let $G^\ddagger = (G^{\widehat{\neg}})^{\widehat{\wedge}_\beta}$. Next set

$$N = \{E \in G^\ddagger \mid |E| < 2^\kappa\}$$

and

$$P = \{E \in G^\ddagger \mid |E| = 2^\kappa\},$$

and note that $G^\ddagger = N \cup P$. Clearly, also, $|P| = 2^\kappa$.

Order P and $\mathscr{R}_{\mathscr{L}}$ each by 2^κ so that $P = \{E_\delta\}_{\delta<2^\kappa}$ and $\mathscr{R}_{\mathscr{L}} = \{v_\xi\}_{\xi<2^\kappa}$. Define the following:

$$\begin{array}{ll} \eta_0 = \mu\nu(v_\nu \in E_0) & \zeta_0 = \mu\nu(\nu > \eta_0, v_\nu \in E_0) \\ L_0 = \{v_{\eta_0}\} & R_0 = \{v_{\zeta_0}\} \\ \eta_\delta = \mu\nu(v_\nu \in E_\delta - \bigcup_{\gamma<\delta} L_\gamma) & \zeta_\delta = \mu\nu(v_\nu > \eta_\delta, v_\nu \in E_\delta - \bigcup_{\gamma<\delta} L_\gamma) \\ L = \bigcup_{\delta<2^\kappa} L_\delta & R = \bigcup_{\delta<2^\kappa} R_\delta \end{array}$$

Note that $|L| = |R| = 2^\kappa$.

We claim that $L \notin ((G^{\widehat{\neg}})^{\widehat{\wedge}_\beta})^{\widehat{\vee}_\beta}$. Suppose the contrary. If $L = \widehat{\vee}_\beta(u)$ with $u \in {}^{\leq\beta}N$, then by the regularity if 2^κ, $|L| \leq \beta \cdot \sup_{n\in ran(u)} |u| < 2^\kappa = |L|$, a contradiction. So by Proposition 35, $E = \widehat{\vee}_\beta(u)$, with $E_{\delta_0} \in ran(u) \cap P$ for some $\delta_0 < \beta$. But then $v_{\zeta_{\delta_0}} \in E_{\delta_0}$ (by the definitions of ζ_{δ_0}). Thus, $v_{\zeta_{\delta_0}} \in L$, since $L = \widehat{\vee}_\beta(u)$ and $E_{\delta_0} \in ran(u)$. This contradicts the fact that $L \cap R = \emptyset$, because $v_{\zeta_{\delta_0}} \in R$.[9]

However, since $|L| = |\mathscr{R}_{\mathscr{L}} - L| = 2^\kappa$ (because $R \subset \mathscr{R}_{\mathscr{L}}$), by Proposition 20, $L \in \bigcup IND_\lambda[\kappa]$. $\square$

Theorem 34 follows immediately from Propositions 35, 36, and 39. Furthermore, via the usual truth-functional tinkering, we have the following.

Corollary 40. *If 2^κ is regular, then for any $G \in IND_\lambda[\kappa]$, $\bigcup IND_\lambda[\kappa] \not\subseteq G^{\widehat{N}_\beta}$.*

References

[1] G.E.M. Anscombe. *An Introduction to Wittgenstein's* Tractatus. Thoemmes Press, Chippenham, Wiltshire (UK), 1971 (1996 reprint).

[2] Thomas Jech. *Set Theory.* Springer-Verlag, New York, third millenium edition, 2002.

[3] Ludwig Wittgenstein. *Tractatus-Logico Philosophicus.* Routledge, New York, 1974.

[9] This construction, by a similar argument, also establishes that $\widehat{\neg}(z) \notin G^\dagger$. However, this is unnecessary because of the Proposition 36.